MATH 212

Assignment 3 /50

2.4.1 – 2.4.3 - 2.4.4 (a,b,c,e) – 2.4.6 – 2.4.11 (a,b) -
2.4.19 (a).

2.4.1- $u(t,x) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$

4 where $f(x) = e^{-x^2}$, $g(x) = \sin x$

$$\Rightarrow u(t,x) = \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s\,ds$$

$$= \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right] - \frac{1}{2c}\cos s\Big]_{x-ct}^{x+ct}$$

$$= \frac{1}{2}\left[e^{-(x+ct)^2} + e^{-(x-ct)^2}\right] - \frac{1}{2c}\left[\cos(x+ct) - \cos(x-ct)\right]$$

2.4.3- $u_t(0,x) = \begin{cases} 1 & 1<x<2 \\ 0 & \text{otherwise} \end{cases} = g(x)$

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$u(0,x) = 0 = f(x)$ $\qquad c=1$

$$u(t,x) = \frac{1}{2}\left[f(x+t) + f(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

$$= \frac{1}{2}\left[0+0\right] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds = \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

* if $x-t \le 1$ and $x+t \le 1$

$$\Rightarrow u(t,x) = \frac{1}{2}\int_{x-t}^{x+t} 0\,ds = 0$$

* if $x-t \le 1$ and $1 < x+t < 2$

$$\Rightarrow u(t,x) = \frac{1}{2}\left[\int_{x-t}^{1} 0\,ds + \int_{1}^{x+t} ds\right] = \frac{1}{2}\left[x+t-1\right]$$

* if $x-t \le 1$ and $x+t \ge 2$

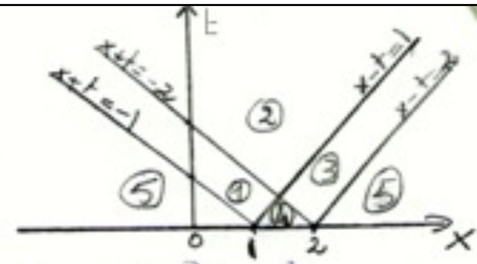
$$\Rightarrow u(t,x) = \frac{1}{2}\left[\int_{x-t}^{1} 0\,ds + \int_{1}^{2} ds + \int_{2}^{x+t} 0\,ds\right] = \frac{1}{2}\left[2-1\right] = \frac{1}{2}$$

* if $x-t > 1$ and $x+t \ge 2$

$$\Rightarrow u(t,x) = \frac{1}{2}\left[\int_{x-t}^{2} ds + \int_{2}^{x+t} 0\,ds\right] = \frac{1}{2}\left[2-x+t\right]$$

* if $x-t \geq 2$ and $x+t \geq 2$

$$\Rightarrow u(t,x) = \frac{1}{2}\left[\int_{x-t}^{x+t} 0\, ds\right] = 0$$



* if $x-t > 1$ and $x+t < 2$

$$\Rightarrow u(t,x) = \frac{1}{2}\left[\int_{x-t}^{x+t} ds\right] = \frac{1}{2}\left[x+t-x+t\right] = t$$

$$\therefore u(t,x) = \begin{cases} \frac{1}{2}(x+t-1) & x-t \leq 1,\ 1 < x+t < 2 \ ① \\ \frac{1}{2} & x-t \leq 1,\ x+t \geq 2 \ ② \\ \frac{1}{2}(2-x+t) & 2 > x-t > 1,\ x+t \geq 2 \ ③ \\ t & x-t > 1,\ x+t < 2 \ ④ \\ 0 & \text{otherwise} \ ⑤ \end{cases}$$

2.4.4. a) $\cos x \cos t = \dfrac{\cos(x+t) + \cos(x-t)}{2}$

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$u(0,x) = \cos x$

$u_t(0,x) = 0$

b) $\cos 2x \sin 2t = \dfrac{\sin(2t+2x) + \sin(2t-2x)}{2}$

$u(0,x) = 0$

$u_t(0,x) = 2\cos 2x$

c) $e^{x+t} = \dfrac{e^{x+t} + e^{x-t}}{2} + \dfrac{e^{x+t} - e^{x-t}}{2}$

$u(0,x) = e^x = f(x)$

$u_t(0,x) = e^x = g(x)$

e) $u(0,x) = 0 = f(x)$

$u_t(0,x) = 3x^2 = g(x)$

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} 3s^2\, ds = s^3\Big]_{x-t}^{x+t}$$

$$= \frac{1}{2}\left[(x+t)^3 - (x-t)^3\right] = \left[\cancel{x^3} + t^3 + 3x^2t + \cancel{3xt^2} - \cancel{x^3} + t^3 + 3x^2t - \cancel{3xt^2}\right]\frac{1}{2}$$

$$= t^3 + 3x^2t$$

2.4.6- Given that $u(t,x)$ is sol of $\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,x) = f(x) \\ u_t(0,x) = g(x) \end{cases}$

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Show that $v(t,x) = u(t-t_0, x)$ is sol to

$$\begin{cases} v_{tt} = c^2 v_{xx} \\ v(t_0,x) = f(x) \\ v_t(t_0,x) = g(x) \end{cases}$$

$V_{tt} = u_{tt}(t-t_0,x)$ by Chain rule
$V_{xx} = u_{xx}(t-t_0,x)$

Since $u_{tt}(t,x) = c^2 u_{xx}(t,x) \quad \forall t>0$

then $u_{tt}(t-t_0,x) = c^2 u_{xx}(t-t_0,x) \quad \forall t>t_0$

$$V_{tt} = c^2 V_{xx} \quad 4$$

So $v(t,x) = u(t-t_0,x)$ is a sol to $v_{tt} = c^2 v_{xx}$ $\forall t>t_0$

Then $v(t_0,x) = u(t_0-t_0, x) = u(0,x) = f(x)$ 1.5

$v_t(t_0,x) = u_t(t_0-t_0,x) = u_t(0,x) = g(x)$ 1.5

2.4.11- a) $c^2 = 4 \Rightarrow c = 2$

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$f(x) = \sin x$

$g(x) = \cos x$

$$u(t,x) = \tfrac{1}{2}[f(x+2t) + f(x-2t)] + \tfrac{1}{4}\int_{x-2t}^{x+2t} g(s)\,ds$$

$$= \tfrac{1}{2}[\sin(x+2t) + \sin(x-2t)] + \tfrac{1}{4} \sin x \Big]_{x-2t}^{x+2t}$$

$$= \tfrac{1}{2}[\sin(x+2t) + \sin(x-2t)] + \tfrac{1}{4}[\sin(x+2t) - \sin(x-2t)]$$

$$= \tfrac{3}{4}\sin(x+2t) + \tfrac{1}{4}\sin(x-2t) \quad 4$$

b) True. $u(t+\pi, x) = u(t,x)$ 2

2.4.19- a) 15